The Hitchhiker's Guide to the Galaxy of Mathematical Tools for Shape Analysis

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Abstract

This course is meant as a practical mathematical guide for researchers and practitioners who are willing to explore the new frontiers of 3D shape analysis, and thus require to manage the rather complex mathematical tools most methods rely on. The target audience includes therefore academia as well as industries or companies active in the shape analysis area. The attendees will familiarize with basic concepts in Differential Geometry, and proceed to advanced notions of Algebraic Topology, always keeping an eye on computational counterparts. The attendees will be shown how these notions can be transferred to practical solutions, through examples of applications to shape correspondence, symmetry detection, and shape retrieval.

The main reason for proposing a comprehensive (yet concise) mathematical guide is that a number of research solutions come from advances in pure and applied Mathematics, as well as from the re-reading of classical theories and their adaptation to the discrete setting. Being able to manage such complex mathematical tools is key to understanding and orienting among the growing number of different proposals. In a world where disciplines (fortunately) have blurred boundaries, we also believe this guide will give some advice on how to make mathematicians, other scientists and practitioners get along well with each other, that is, how to talk to each other – and get to *understand* each other. We hope that, at the end of the course, attendees will have an idea on how to find the right mathematical tools that match a bright intuitive idea, and how to strike a balance between being theoretically rigorous and offering computationally feasible solutions... possibly keeping our guide on their desks.

The course is structured as a half-day course. We assume the participants have basic skills in Geometric Modelling and familiarity with basic concepts in Mathematics.

1 About the Lecturers

Silvia Biasotti: Silvia Biasotti is a researcher at at the Institute of Applied Mathematics and Information Technologies (IMATI) of the National Research Council (CNR) of Italy, Research Unit of Genoa, where she works in the Shape Modelling group. She got a Laurea degree in Mathematics in September 1998 from the University of Genova; in May 2004 she got a a PhD in Mathematics and Applications and in April 2008 a PhD in Information and Communication Technologies, both from the University of Genoa. She authored more than 80 scientific peer-reviewed contributions, is a member of the editorial board of ISRN Machine Vision and served in the programme committee of SMI06-SMI11. Her research interests include the study of topological-geometrical descriptions of 2D and 3D models and the development of geometric reasoning techniques for the extraction of shape features from discrete surface models. In the last years, her research interests concerned computational topology techniques for the analysis and structuring of geometric information in any dimension, and shape similarity based on similarity between structures. She is also the proponent and leader of the CNR activity *Topology and Homology for analysing digital shapes* and teaches the Master course *Methods of analysis of discrete surfaces and their applications* at the Dept. of Mathematics, University of Genoa.

Bianca Falcidieno Bianca Falcidieno is a Research Director at the Institute of Applied Mathematics and Information Technologies (IMATI) of the National Research Council (CNR) of Italy. She is the Responsible for the Genova branch of IMATI (RUOS) and the President of the CNR Research Area of Genova. She has been leading and coordinating research at international level in advanced and interdisciplinary fields, including Computational Mathematics, Computer Graphics, Multidimensional Media and Knowledge Technologies. She coordinated various international and national projects, including the EU Network of Excellence AIM@SHAPE (2004-2008), the EU Coordination Action FOCUS K3D (2008-2010), the Italy-Israel project FIRB SHALOM (2006-2009). Bianca Falcidieno is the author of over 200 scientific refereed papers and books. She was in charge of several international commitments, including editorial tasks and the chairing or co-chairing of events such as the IEEE Conference on Shape Modeling International (SMI, France 2010) and the Conference on Semantics and digital Media Technology (SAMT, Italy 2007). She is the editor in chief of the International Journal of Shape Modelling (World Scientific). In her training activity, she supervised several researchers, while taking care of the guidance and training of PhD and master students, both Italians and foreigners, by teaching courses and supervising theses and doctoral activities, both in Italy and abroad, on Applied Mathematics and Information Technologies. For the 80th CNR anniversary, she was included in the 12 top-level female researchers in the CNR history. In

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2011 Bianca Falcidieno was elected to a Fellowship of the EUROGRAPHICS Association in recognition of her scientific contribution to the advancement of Computer Graphics.

Daniela Giorgi: Daniela Giorgi graduated cum laude in Mathematics at the University of Bologna in 2002, with a thesis on geometric modelling of curves and surfaces; she then got a PhD in Computational Mathematics from the University of Padova in 2006, with a thesis on image and 3D model retrieval. She joined the Centre of Excellence ARCES in Bologna and then moved to Genova, where she joined the Shape Modeling Group at IMATI-CNR as a researcher. Her distinguishing features are strong mathematical expertise (Differential Geometry, Morse theory, Topology) together with in-depth knowledge in ICT and computational fields (Computer Graphics, Image and 3D Processing). Her main research interests concern multimedia analysis, description and retrieval. She has been developing . She is the author of over 30 peer-reviewed international publications in high-level journals, books and conferences, about computational geometry and topology tools for shape analysis, description, and retrieval. She participated in several national and international research projects, and was in charge of the Watertight Models Track (2007) and of the Classification of Watertight Models Track (2008) of the SHREC event (SHape REtrieval Contest). She has been teaching BS (Engineering) and Master (Mathematics and its Applications) courses at University; she has supervised trainees, undergraduates and master students, thus achieving education and knowledge transfer competency. She also was a lecturer at International Schools.

Michela Spagnuolo: Michela Spagnuolo got a Laurea Degree cum laude in Applied Mathematics from the University of Genova and a PhD in Computer Science Engineering from the INSA, Lyon. She is currently a Senior Researcher at CNR-IMATI. She authored more than 130 reviewed papers in scientific journals and international conferences, edited a book on 3D shape analysis, and was a guest-editor of several special issues. She is an associate editor of Computers&Graphics, Computer Graphics Forum and The Visual Computer. She is a member of the steering committee of the IEEE Shape Modelling International (SMI), and was the programme chair of the EG and ACM workshops on 3D Object Retrieval (3DOR) and the International Conference on Semantic and Media Technology (SAMT). Her current interests include 3D modelling and visualization, shape analysis techniques, shape similarity and matching, and computational topology. She was responsible for several EC and national projects of CNR-IMATI and is currently responsible for the research unit on *Advanced techniques for the analysis and synthesis of multidimensional media* and for the research unit on Modeling and analysis techniques, and high performance and grid omputing of a CNR Project on Bioinformatics.

2 Course Outline

MODULE 1: Moderator: Bianca Falcidieno
Lecturers: Michela Spagnuolo, Silvia Biasotti.

A. **Introduction and welcome.** (14:00-14:05)

- Overview of the course and motivation.

B. **Mathematics and shape analysis challenges.** (14.05-14.20)

- Shape properties and invariants;
- Similarity between shapes.

C. **Mathematical Guide, Part 1.** (14.20-15.00)

- Topological spaces, functions, manifolds, metric spaces;
- Isometries, geodesics, curvature, Riemann surfaces, Laplace-Beltrami operator;
- Gromov-Hausdorff distances.

D. **Examples of Applications, Part 1.** (15.00-15.30)

- surface correspondence;
- symmetry detection;
- intrinsic shape description.

Break (15.30-15.40)

MODULE 2: Moderator: Michela Spagnuolo
Lecturers: Daniela Giorgi, Bianca Falcidieno.

E. **Mathematical Guide, Part 2.** (15.40-16.20)

- Basics on algebraic topology, simplicial Complexes, Homology, surface genus;
- Critical points, Morse Theory.

F. **Examples of Applications, Part 2.** (16.20-16.50)

- Persistent topology;

- Reeb graphs.

G. **Conclusions** (16.50-17.15)

- Discussion on recent trends and open issues, supported by case studies.

3 Introduction

In the last decade we have witnessed great interest and a wealth of promise in 3D shape analysis, where the goal is to derive geometric, structural and semantic information about 3D objects from low-level properties. While the first half of the decade can be thought of as the initial phase of research, which only laid foundation to such promise, the second half saw a large number of new techniques and systems, and got many new people involved. The community has started to reason on new challenges, including similarity under deformations other than rigid motions, partial matching, correspondence finding, symmetry detection, view-point selection, semantic annotation and attribute transfer. Lateral evolution has also occurred in terms of the associated applied domains, spanning various fields from Medicine to Bioinformatics and Architecture.

These new challenges required more elaborate methods: a number of interesting solutions came from advances in (pure and applied) Mathematics, as well as from the re-reading of classical mathematical theories and their adaptation to the discrete setting. Being able to manage such complex mathematical tools is key to understanding the most recent research solutions, and orienting among the growing number of different proposals. In this scenario, this course is meant as a practical guide to familiarize with most of the mathematical concepts and computational tools that are used in recent work on the analysis of 3D objects, from basic concepts in Differential Geometry to notions of Algebraic Topology. The course includes a summary of the background mathematical notions, a detailed presentation of the mathematical methods underlying recent shape analysis works, and examples of applications to shape correspondence, symmetry detection, shape comparison and retrieval.

3.1 Overview of the Course Material

The course is structured as a half-day course. The fist part introduces some of the main challenges in shape analysis, underlining the key role that Mathematics plays. Then, the first part of the mathematical guide is presented, dealing with concepts mainly in Differential Geometry and Topology; examples are shown about surface correspondence and symmetry detection, to demonstrate how the surveyed mathematical concepts have been exploited in recent research works.

In the second part, the mathematical guide is completed with advanced concepts in Differential Geometry and Algebraic Topology, whose use is demonstrated in shape comparison and retrieval applications. In the concluding part, we will draw some conclusions about the use of Mathematics in shape analysis: with the help of case studies, possibly taken from recent shape analysis contests (e.g., the SHREC 2012 Track on Stability on Abstract Shapes), we shall reason about to what extent it has reached his full potential, and what still has to be done.

The course material is partly based on previously published papers, talks and lectures by the authors. These include:

- the papers published in ACM Computing Surveys [Biasotti et al. 2008b] and in Theoretical Computer Science [Biasotti et al. 2008c] about geometrical-topological tools for shape analysis and description, which covered mathematical, computational and applicative aspects, and both received a good appreciation from the research community;
- the tutorial presented at EUROGRAPHICS 2007 [Biasotti et al. 2007], about shape comparison and retrieval methods rooted in Morse Theory;
- the MiniSymposium *Geometric-topological methods for 3D shape classification and matching*, held at ICIAM (International Council for Industrial and Applied Mathematics) 2007;
- lectures given at international schools (AIM@SHAPE International Summer School on Computational Methods for Shape Modelling and Analysis - 2004; AIM@SHAPE International Summer School on Shape Modeling and Reasoning - 2007; Utrecht Summer School on Multimedia Retrieval - 2007; Seminar on Non-Textual Data Searching Systems (http://diuf.unifr.ch/diva/3emeCycle08) - 2008) and at national events (DIMA Workshop *Matematica, Forme, Immagini* - 2010).

The tutorial will also reflect the many years' experience of organizing the EUROGRAPHICS workshop on 3D Object Retrieval (EG 3DOR), and the launching and contributions to the SHape REtrieval Contest (SHREC): launched by the AIM@SHAPE project in 2006, SHREC has seen an increasing participation of researchers, and evolved into a multi-contest featuring diverse tracks on 3D retrieval, correspondence finding, shape segmentation and related topics (http://www.aimatshape.net/event/SHREC). This experience will allow us to demonstrate and benchmark recent results, and not just to describe them theoretically.

3.2 Educational Role

The notes are mostly aimed at researchers who are willing to explore the new frontiers of 3D shape analysis, and thus require to manage the rather complex mathematical tools which most methods rely on. We assume that the participants have basic skills in Geometric Modelling, and familiarity with basic concepts in Mathematics. The educational target is ambitious, in that it requires to strike an happy medium between complex (and vast) mathematical theories, computational aspects, and practical issues. Our mission is to offer a comprehensive yet concise mathematical guide, which can help a new generation of researchers to truly understand what is behind the most recent solutions in shape analysis.

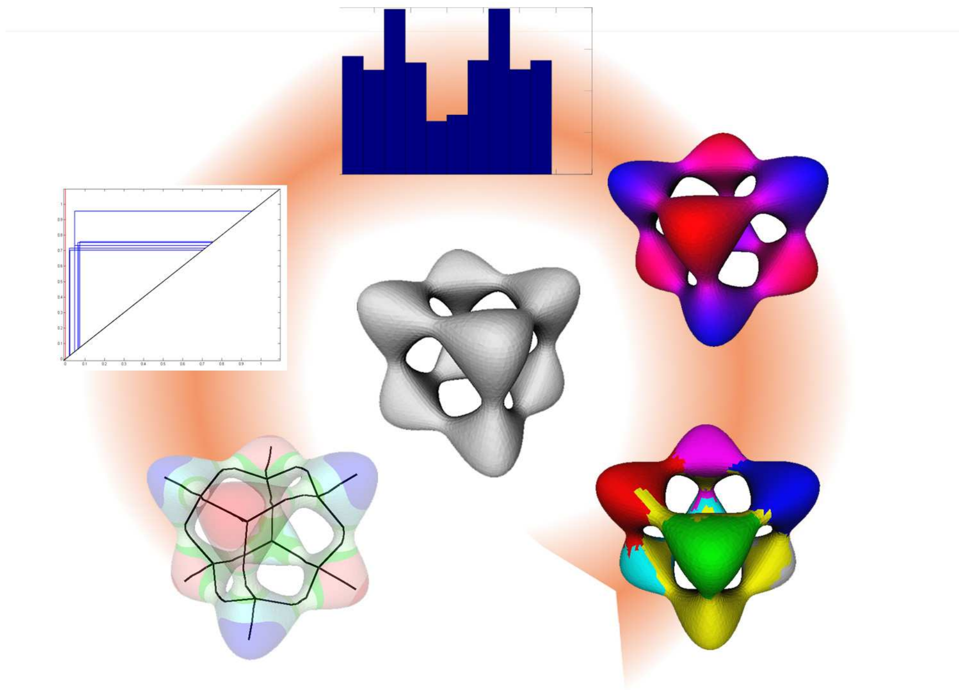

Figure 1: *Mathematics, shapes, invariants and descriptors.*

Previous SIGGRAPH courses covered topics in Mathematics and Discrete Mathematics (including the 2006 course on Discrete Differential Geometry: An applied introduction; Surface Modeling and Parametrization with Manifolds, and Manifolds and modeling - 2005; Geometric signal processing on large polygonal meshes - 2001), but a comprehensive course collecting the mathematical background pertaining to different fields in advanced shape analysis, and spanning from basics in Differential Geometry to Algebraic Topology, has not been proposed yet. Moreover, existing surveys on shape analysis [Tangelder and Veltkamp 2008; van Kaick et al. 2011] do not cover the Mathematics behind the research solutions surveyed. We believe it timely to fill the gap and visit this complex material, with the aim of helping a good understanding of novel, complex research solutions, and their transfer into practical applications.

4 Contents

In many problems in Computer Graphics, it is convenient to model shapes as topological spaces, possibly manifolds; often, shape data are endowed with a notion of distance between their points, which turns them, in the language of Differential Geometry, into metric spaces. Capturing the information contained in shape data thus typically takes the form of computing shape properties, and turning them into invariants, or signatures, which provide insights about the shape characteristics. Measuring shape properties (distances between points, curvature, etc.) and getting invariants is a fundamental problem in Computer Graphics, which has applications to correspondence finding, symmetry detection, and more.

A more elaborate question concerns the definition of distances between shapes. Indeed, one of the cornerstone problems in shape analysis is how to define a notion of shape (dis)similarity; that is, we may want to analyze to what extent two spaces represent two instances of some common class, up to a certain notion of invariance. Having defined a proper notion of distances between shapes, it is natural to ask for shape descriptors which are able to signal shape (dis)similarity in accordance with this definition. This has fundamental applications in shape matching, recognition and retrieval.

In what follows, we expand on these challenges, point out why (and what) Mathematics is needed to make our way through complex shape analysis problems, and list the concepts we will present in our tutorial.

4.1 Computing 3D shape properties and metric invariants

When we think about shape properties, the first distinction to be made is between extrinsic and intrinsic shape properties. *Extrinsic* properties are the properties related to how the shape is laid out in the Euclidean 3D space. If we model a shape as a *metric space*, its extrinsic properties can be described by using the Euclidean distance between points. Euclidean distances form the basis for most of the earliest shape analysis methods in Computer Vision and Computer Graphics. At the same time, lately the study of *intrinsic* properties, that is, properties related to the metric structure and invariant to shape deformations, started penetrating into the Vision and Graphics communities. The reason is that deformable objects are ubiquitous in our reality, from human organs to living beings. If a shape is modeled as a metric space, intrinsic properties can be described using *geodesic distances*, which, on a surface, measure the length of the shortest path along the surface between two points. The use of geodesic distances proved effective in a number of studies, and paved the road to a number of tools for intrinsic non-rigid shape analysis. Recent developments include the introduction of *fuzzy geodesics*, which relax the notion of shortest path so as to increase robustness; *diffusion distances* (and related notions such as *biharmonic distances* and the *heat kernel*), which are related to the physical process of heat diffusion on a surface from a source point; *inner distances* and *interior distances* to be computed on volumes. Concerning surface properties and invariants, a fundamental concept is the *Gaussian curvature*, with the peculiarity that it depends on the metric defined on the space (different metrics induce different curvatures), whereas the total curvature only depends on the space topology.

If we stick to the metric space model, we can see how distances between points can originate distances between spaces. Well known distances are the *Hausdorff distance*, which measures how far two subsets of a metric space are from each other, and the *Wasserstein metric*, defined between probability distributions on metric spaces. Another interesting example is the *Gromov-Hausdorff* distance, which casts the comparison of two spaces as a problem of comparing pairwise distances on the spaces. Equivalently, the computation of the Gromov-Hausdorff distance between spaces can be posed as measuring the distortion caused by embedding one metric space into another, that is, evaluating how much the metric structure is preserved while mapping a shape into the other. By considering different metrics between points, we get different notions of metrics between spaces [Gromov et al. 2006; Bronstein et al. 2010].

Mathematics gives the whats and whys. From the mathematical point of view, understanding and managing all the concepts listed above require a background in Differential Geometry and Topology [do Carmo 1976; Guillemin and Pollack 1974; Hirsch 1997]. We will discover how to model a shape as a *topological space* and a *metric space*, what *(Riemannian) manifolds* are useful for, the precise definitions of widely used terms such as *geodesic*, *isometry*, *curvature*, and how they relate to *conformal geometry* and the highly-cited *Laplace-Beltrami operator* [Jost 2005; Reuter et al. 2009; Zeng et al. 2010]. We will see how these notions are fundamental to analyse shape properties and compute shape invariants. Having this background in mind, we will analyze all notions of surface properties and metric invariants listed above, from the theoretical and the computational point of view.

The how-to in applications: surface correspondence, symmetry detection and intrinsic shape description. At this point, we will be able to show how the surveyed concepts were applied to solve different problems, namely symmetry detection, surface correspondence and intrinsic shape description. Concerning symmetry detection, we will refer to [Kim et al. 2010], where geodesics distances and conformal mappings are used to generate symmetry invariant point sets and detect surface self-isometries, that is, intrinsic symmetries. Concerning surface correspondence, reference works will be [Lipman and Funkhouser 2009], where differential and conformal geometry give rise to a voting scheme that identifies corresponding points which are consistent with isometric mappings of large surface regions, and [Sun et al. 2009], where diffusion geometry and the Heat Kernel Signature are used to detect repeated structure within the same shape and across a collection of shapes.

4.2 The mathematical notion of similarity between shapes, and the role of shape descriptors

If we push further the idea of measuring the distortion of properties while transforming a shape into another, we get the concept behind the *Natural pseudo-distance*. Let us assume now that a shape is a space endowed with a real function, which describes some shape properties. To compare two shapes, we can imagine to transform one shape into the other, and check how much the properties of the original shapes have been preserved/distorted; this amounts to measure how much the values of the real function representing those properties have been altered. The Natural pseudo-distance offers a framework in which we can plug-in different properties, in the form of different real functions, so as to measure shape (dis)similarity up to different notions of invariance.

Having defined a proper notion of distances between shapes, the problem has been addressed of defining shape descriptors which are stable under perturbations of the shape defined in the distance space. These descriptors include *size functions*, which have been proven to be stable under the natural pseudo-distance, and the family of *persistent homology* tools. These signatures are able to naturally combine the classifying power of topology with the descriptive power of geometry, and have a close relation with other popular tools such as *Reeb graphs*, which have their roots in the same theoretical settings.

Mathematics gives the whats and whys. At this point, we will need to further explore the mysteries of mappings between topological spaces, that is, the notions of *homeomorphisms* and *diffeomorphisms* between topological spaces [Griffiths 1976; Fomenko 1995]. Basic notions of Algebraic Topology will have to be introduced, starting from the notion of *simplicial complexes*, and going through *homology* [Willard 1970; Engelking and Sielucki 1992; Massey 1967; Hatcher 2001]. We will see how *Morse Theory* elegantly bridges geometrical properties of shapes with their topology [Milnor 1963; Edelsbrunner and Harer 2008]. Having this background in mind, we will show how all these mathematical concepts form the basis for the definition of distances between shapes (e.g. the Natural pseudo-distance [Frosini and Mulazzani 1999; Donatini and Frosini 2007]), and the computation of shape descriptors as those listed above (size functions, persistence diagrams, Reeb graphs).

The how-to in applications: We will overview shape description at the light of the persistent topology framework, with particular attention to persistent homology [Edelsbrunner and Harer 2008] and barcodes [Ghrist 2008]. Then, we will introduce size theory [Frosini 1990; Frosini 1991; Frosini and Landi 2001] and consider [Biasotti et al. 2008a; Biasotti et al. 2011], which use persistent topology and multidimensional size functions for retrieving 3D objects in database, according to different similarity criteria and invariance concepts. Finally, we will overview the use of Reeb graphs [Reeb 1946] in the shape analysis, description and retrieval arena [Shinagawa et al. 1991; Hilaga et al. 2001; Dey and Wang 2011].

4.3 Conclusions

At the end of the course, some case studies taken from recent shape analysis contests (e.g., the SHREC 2012 Track on Stability on Abstract Shapes) will offer us the possibility of further reasoning on what Mathematics has done and still can do for shape analysis.

As Mathematicians doing research in a world where disciplines (fortunately) have blurred boundaries, we will also try to give some advice on how to make mathematicians and other scientists get on well with each other, that is, how to talk to each other – and get to *understand* each other. We hope that, at the end of the course, attendees will have an idea on how to find the right mathematical tools that match a bright intuitive idea, and how to strike a balance between being theoretically rigorous and offering computationally feasible solutions... possibly keeping our guide on their desks.

Acknowledgements

This work is partially supported by the projects: *VISIONAIR: Vision Advanced Infrastructure for Research*, European project FP7 INFRASTRUCTURES, 2011-2015 and *MULTISCALEHUMAN: Multi-scale Biological Modalities for Physiological Human Articulation*, European project FP7 PEOPLE Initial Training Network, 2011-2014. In addition, the lectures thank P. Frosini, M. Ferri and the Vision Mathematics group of the Univ. of Bologna and C. Landi of the Univ. of Modena and Reggio Emilia for the helpful discussions and hints.

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The Hitchhiker's Guide to the Galaxy of Mathematical Tools for Shape Analysis

SIGGRAPH 2012 Course Notes

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outline

- ✓ motivation
- ✓ mathematics and shape analysis challenges
  - shape properties and invariants
  - similarity between shapes
- ✓ mathematical guide (Part 1)
  - topological spaces, functions, manifolds
  - metric spaces, isometries, geodesics, curvature
  - Gromov-Hausdorff distance
- ✓ concepts in action (Part 1)
  - symmetry detection
  - surface correspondence
  - shape characterization

outline

- ✓ mathematical guide (Part 2)
  - simplicial complexes
  - basics on algebraic topology and homology
  - Morse theory
  - natural pseudo-distance
- ✓ concepts in Action (Part 2)
  - persistent topology
  - Reeb graphs
- ✓ discussions and trends
- ✓ conclusions

where are we now?

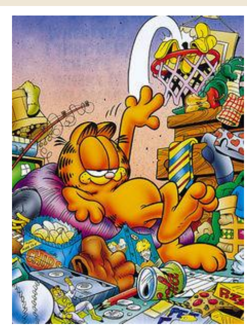
- ✓ technology today
  - hardware for visualizing 3D and 3D acquisition technologies: "3D on the desktop"
  - computer networks: fast connection, low cost
  - 3D printers: not only mock-ups but even end products
  - …

rendering, acquiring, transmitting, "materializing" 3D content is now feasible in specialized as well as unspecialized contexts

3D media

- ✓ **non professionals**
  - 3D social networking
  - "broad semantic context"
- ✓ **professionals**
  - Product Modeling
  - Design
  - Cultural Heritage
  - Gaming
  - Simulation
  - Medicine
  - Bioinformatics
  - Architecture
  - Archeology
  - …

… how to analyse, describe, process, organize, navigate, filter, share, re-use and re-purpose, this large amount of complex content ?



The Hitchhiker's Guide to the Galaxy of Mathematical Tools for Shape Analysis

SIGGRAPH 2012 *Course Notes*

Mathematics and shape analysis challenges

---

what does "shape" mean?

- "…all the geometrical information that remains when location, scale, and rotational effects (Euclidean transformations) are filtered out from an object" [Kendall 1977]

…uhmmm… NOT sure about this…

---

what does "shape" mean?

- "…the form of something by which it can be seen (or felt) different by something else" [Longman Dictionary of Contemporary English]

That sounds nice but… what do "similar" and "different" mean?
It seems like a chicken and egg situation…

---

shape, similarity & the observer

- things possess a shape for the observer, in whose mind the association between the perception and the existing conceptual models takes place [Koenderink 1990]

- similarity is a cognitive process, depending on the observer and the context

---

shape and view points

Impossible Ring and Pillars

Guido Moretti's sculptures

---

different flavours

different flavours

geometric congruence

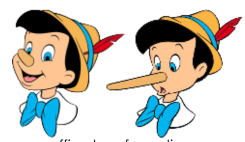
structural equivalence

Functional equivalence

“class” equivalence

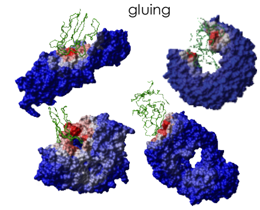
different flavours

Isometric transformation



affine transformation

elastic deformations and gluing



intuition vs mathematics

- ✓ congruence
  - two objects are congruent if one can be transformed into the other by rigid movements (translation, rotation, reflection – not scaling)

intuition vs mathematics

- ✓ similarity:
  - two geometrical objects are called *similar* if one can be obtained by the other by uniform stretching . Formally, a *similarity* of a Euclidean space S is a function $f: S \rightarrow S$ that multiplies all distances by the same positive scalar *r*, so that:

$$d\big(f(x), f(y)\big) = r d(x, y), \forall x, y \in S$$

intuition vs mathematics

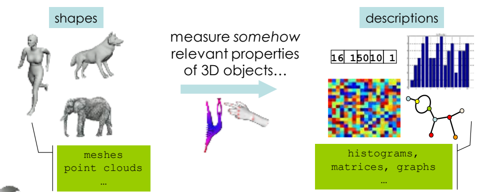
- ✓ an **affine transformation** is a deformation that map straight lines into straight lines
  - it doesn't respect lengths or angles
  - it preserves all affine combinations (i.e., linear combinations in which the sum of the coefficients is 1)

mathematics: shape description and similarity

- ✓ similar shapes with **respect to what**?
  - shape descriptions, to code the aspects of shapes to be taken into account and manage the complexity of the problem
- ✓ **similarity** in what sense ?
  - transformations among the shapes that we consider irrelevant to the assessment of the similarity
    - invariants or properties

shape and description

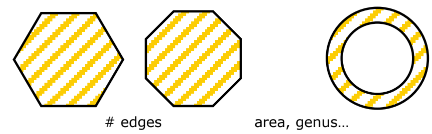
- shape descriptions reduce the complexity of the representation; their choice depends on
  - **type** of shapes and their variability/complexity
  - **invariants** or properties



shape descriptions

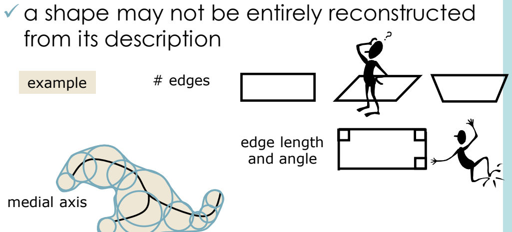
- in general, a description could be just a set of numbers…

example



shape descriptions

- different shapes should have different descriptions
  - different enough to discriminate among shapes
- a shape may not be entirely reconstructed from its description

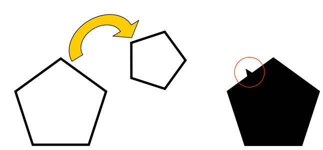


shape descriptions - properties

- invariance
- uniqueness
- stability to noise
- sensitivity to global/local features

example

edges



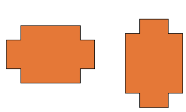
invariance

- invariance = the descriptor does not change for a given object under a class of transformations
- a property P is invariant to transformation T applied to an object O iff

$$P(T(O)) = T(P(O))$$

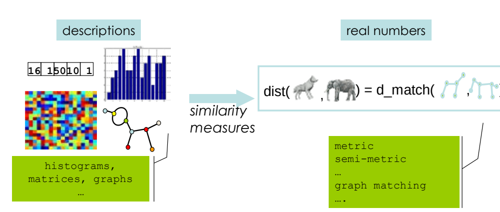
example

boundary length



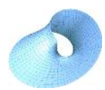
shape descriptions and similarity

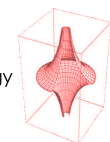
- *similarity* in what sense ?
  - defining appropriate *similarity measures* between shape descriptions



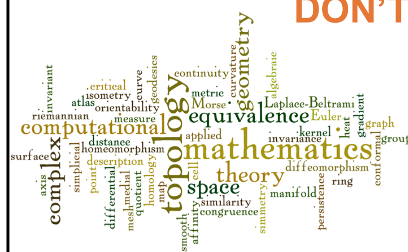
things are not that easy…

- the simple examples we have shown are clearly not enough to deal with the *complexity* at hand… this is where Mathematics comes into play!
- what tools?
  - topological spaces
  - Riemannian surfaces and metrics
  - distances and measures
  - algebraic topology
  - differential geometry and topology
  - Morse theory





DON'T PANIC



The Hitchhiker's Guide to the Galaxy of Mathematical Tools for Shape Analysis

SIGGRAPH 2012 *Course Notes*

Mathematical Guide (part 1)

content

- mathematical concepts
  - topological spaces
  - homeo- and diffeomorphisms
  - manifolds
  - metric spaces
  - geodesic distances
  - Riemannian surfaces
  - curvature
  - Laplace-Beltrami operator
  - Gromov-Hausdorff distance
- concepts in action
  - surface correspondence
  - symmetry detection
  - intrinsic shape description

content

- mathematical concepts
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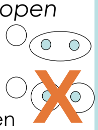
topological space

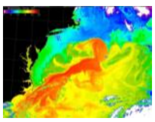
- a *topological space* is a set X together with a collection T of subsets of X, called *open sets,* satisfying the following axioms:
  - $X, \emptyset \in T$
  - any union of open sets is open
  - any finite intersection of open sets is open



- the collection T is called a *topology* on X
- why topological spaces?
  - to represent the set of observations made by the observer (e.g., boundary, interior, projection, contour);
  - to reason about stability and robustness

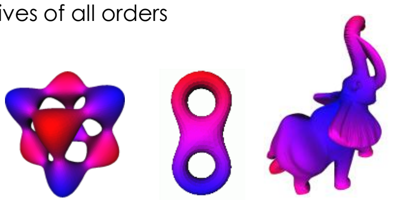
continuous function

- let X, Y topological spaces an arbitrary subset of $\mathbb{R}^n$
  - $f : X \rightarrow Y$ is *continuous* if for every open set $V \subseteq Y$ the inverse image $f^{-1}(V)$ is an open subset of X
- why functions?
  - to characterize shapes
  - to measure shape properties
  - to model what the observer is looking at
  - to reason about stability
  - to define relationships *(e.g., distances)*



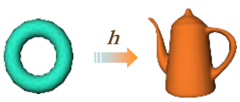
smooth function

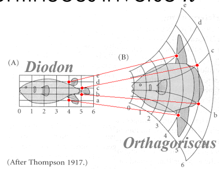
- let X be an arbitrary subset of $\mathbb{R}^n$; $f : X \rightarrow \mathbb{R}^m$ is called *smooth* if $\forall x \in X$ there is an open set $U \subseteq \mathbb{R}^n$ and a function $F: U \rightarrow \mathbb{R}^m$ such that $F = f_{|X}$ on $X \cap U$ and F has continuous partial derivatives of all orders



homeo- & diffeo- morphisms

- a *homeomorphism* between two topological spaces X and Y is a continuous bijection $h: X \to Y$ with continuous inverse h^{-1}





- given $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, if the smooth function $f: X \to Y$ is bijective and f^{-1} is also smooth, the function f is a *diffeomorphism*

manifold

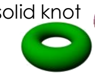
- manifold without boundary
  - a topological Haussdorff space M is called a *k-dimensional topological manifold* if each point $m \in M$ admits a neighborhood $U_i \subseteq M$ homeomorphic to the open disk $D^k = \{x \in \mathbb{R}^k \,||x| < 1\}$ and $M = \cup_{i \in \mathbb{N}} U_i$
- manifold with boundary
  - a topological Haussdorff space S is called a *k-dimensional topological manifold with boundary* if each point $m \in M$ admits a neighborhood $U_i \subseteq M$ homeomorphic either to the open disk $D^k = \{x \in \mathbb{R}^k \,||x| < 1\}$ or the open half-space $\mathbb{R}^{k-1} \times \{y \in \mathbb{R} \mid y \geq 0\}$ and $M = \cup_{i \in \mathbb{N}} U_i$
- k is called the dimension of the manifold

smoothness and orientability

- transition functions
  - let $\{(U_i, \varphi_i)\}$ an union of charts on a k-dimensional manifold M, with $\varphi_i: U_i \to D^k$.
    the homeomorphisms $\varphi_{i,j}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ such that $\varphi_{i,j} = \varphi_j \cap {\varphi_i}^{-1}$ are called *transition functions*
- smooth manifold
  - a k-dimensional topological manifold with (resp. without) boundary is called a *smooth manifold* with (resp. without) boundary, if all transition functions $\varphi_{i,j}$ are smooth
- orientability
  - a manifold M is called orientable is there exists an atlas $\{(U_i, \varphi_i)\}$ on it such that the Jacobian of all transition functions is positive for all intersecting pairs of regions

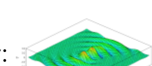
examples

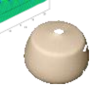
- 3-manifolds with boundary:
  - a solid sphere, a solid torus, a solid knot



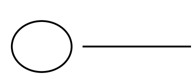
- 2-manifolds:
  - a sphere, a torus

- 2-manifold with boundary:
  - a sphere with 2 holes, single-valued functions (scalar fields)





- 1 manifold:
  - a circle, a line



parametric representation of surfaces

- regular parameterisation of a surface:

$$\Phi: U \subset \mathbb{R}^2 \to \mathbb{R}^3$$

  - $\Phi(u,v) \in S, for\ all\ (u,v) \in U$
  - $\Phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$

such that

$$\begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} \times \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

metric space

- a *metric space* is a set where a notion of distance (called a metric) between elements of the set is defined

- formally,
  - a **metric space** is an ordered pair (S, d) where S is a set and d is a metric on S (also called distance function), i.e., a function

$$d: S \times S \to \mathbb{R}$$

such that $\forall x, y, z \in S$:

- $d(x,y) \geq 0;$ *(non-negative)*
- $d(x,y) = 0$ iff $x = y;$ *(identity)*
- $d(x,y) = d(y,x);$ *(symmetry)*
- $d(x,z) \leq d(x,y) + d(y,z)$ *(triangle inequality)*

spaces and properties

- an isometry is a bijective map between metric spaces that preserves distances, that is

$$f: X \to Y, d_Y\big(f(x_1), f(x_2)\big) = d_X(x_1, x_2)$$

- looking for the right metric space…
  - the Euclidean distance $d(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{n}\sqrt{(q_i - p_i)^2}$
  - geodesic distances, diffusion distances, …

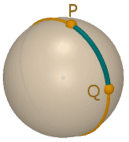
geodesic distance

- given ϕ a regular surface parametrization, the first fundamental form is defined as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

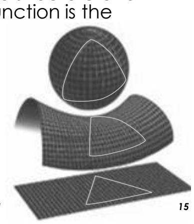
with $E = \left|\frac{\partial\Phi}{\partial u}\right|^2$, $F = \frac{\partial\Phi}{\partial u}\cdot\frac{\partial\Phi}{\partial v}$, $G = \left|\frac{\partial\Phi}{\partial v}\right|^2$

- the arc length of a curve γ is given by $\int_\gamma ds$
- *minimal geodesics*: shortest path between two points on the surface
- *geodesic distance* between P and Q: length of the shortest path between P and Q
- geodesic distances satisfy all the requirements for a metric, including the triangle inequality



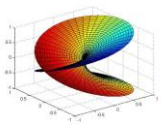
Riemannian surfaces

- a conformal structure is an atlas of the surface such that angles among tangent vectors can be coherently defined on different local coordinate systems
- a surface with a conformal structure is called a Riemann surface
- a Riemannian surface carries the structure of a metric space whose distance function is the geodesic distance
- *(uniformization)* any simply connected Riemann surface is either conformally equivalent to:
  - the open unit disk
  - the complex plane
  - the Riemann sphere



Riemannian surfaces

- a Riemann surface is a complex manifold of complex dimension one
  - a 2-manifold (real) can be turned into a Riemannian surface iff it is orientable and metrizable
  - as a consequence Mobius strip, Klein bottle, projective plan don't admit a conformal structure

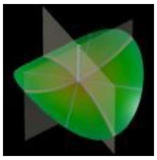


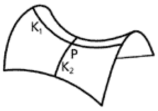
invariance and isometries

- a property invariant under isometries with respect to a Riemannian metric is called an intrinsic property
- examples:
  - the first fundamental form
  - the Gaussian curvature K
  - the geodesic distance
  - the Laplacian operator

principal curvatures

- the principal curvatures measure the maximum and minimum bending of a surface at each point along lines defined by the intersection of the surface with planes containing the normal

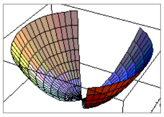




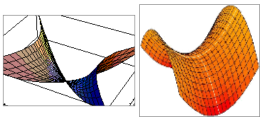
Gaussian and mean curvature

- given k_1 and k_2 the principal curvatures at a point surface
  - Gaussian curvature $K = k_1 k_2$
  - mean curvature $H = (k_1 + k_2)/2$
- according to the behavior of the sign of K, the points of a surface may be classified as
  - elliptic
  - hyperbolic
  - parabolic or planar

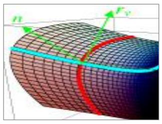
examples



K>0



K<0



$K = 0, H \neq 0$

Laplace-Beltrami operator

- continuous case

$$\Delta f := \mathrm{div}(\mathrm{grad} f)$$

- Laplacian eigenvalue problem

$$\Delta f = -\lambda f$$

- orthonormal eigensystem

$$\mathcal{B} := \{(\lambda_i, \psi_i)\}_i \qquad \Delta \psi_i = \lambda_i \psi_i$$

$$\lambda_0 \leq \lambda_1 \leq \ldots, \lambda_i \leq \lambda_{i+1} \ldots \leq +\infty$$

discrete Laplacian operator

$$\Delta f(\mathbf{p}_i) := \frac{1}{d_i} \sum_{j \in N(i)} w_{ij} \left[f(\mathbf{p}_i) - f(\mathbf{p}_j) \right]$$

- $N(i)$ index set of 1-ring of vertex $\mathbf{p}_i$
- $f(\mathbf{p}_i)$ function value at vertex $\mathbf{p}_i$
- d_i mass associated with vertex $\mathbf{p}_i$
- w_{ij} edge weights

discrete geometric Laplacian

- Desbrun et al. (1999)

$$w_{ij} := \frac{\cot(\alpha_{ij}) + \cot(\beta_{ij})}{2} \qquad d_i := a(i)/3$$

  - the cotangent weights take into account the angles opposite to edges, the masses take into account the area around vertices
- Meyer et al. (2002)

$$d_i := a_V(i)$$

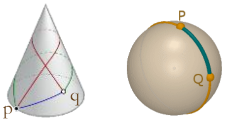
  - cotangent weights, masses considering the Voronoi area
- Belkin et al. (2003, 2008)
  - weights constructed using heat kernels
- Reuter et at. (2005, 2006)
  - weak formulation of the eigenvalue problem

$$\langle \Delta f, \varphi_i \rangle_{\mathcal{L}^2(\mathcal{M})} = -\lambda \langle f, \varphi_i \rangle_{\mathcal{L}^2(\mathcal{M})}$$

with φ_i cubic form functions

metrics between spaces

- from distances between points to distances between metric spaces
- the Gromov-Hausdorff distance poses the comparison of two spaces as the direct comparison of pairwise distances on the spaces; equivalently, it measures the distortion of embedding one metric space into another



Gromov-Hausdorff distance

- let $(X;\ d_X), (Y; d_Y)$ be two metric spaces and $C \subset X \times Y$ a correspondence
- the distortion of C is:

$$dis(C) = \sup_{(x,y),(x',y')\in C} |d_X(x,x') - d_Y(y,y')|$$

- the Gromov-Hausdorff distance is

$$d_{GH}(X,Y) = \frac{1}{2}\inf_{C} dis(C)$$

- variations: *Lp Gromov-Hausdorff* distances and *Gromov-Wasserstein* distances

properties

- the Gromov-Hausdorff distance is parametric with respect to the choice of metrics on the spaces X and Y
- common choices
  - Euclidean distance (estrinsic geometry)
  - geodesic distance (intrinsic geometry) or, alternatively, diffusion distance

$$d^2{}_{X,t}(x,y) = \sum_{i=0}^{\infty} e^{-2\lambda_i t}(\psi_i(x) - \psi_i(y))^2$$

where (λ_i, ψ_i) is the eigensystem of the Laplacian operator and t is time

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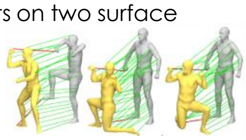
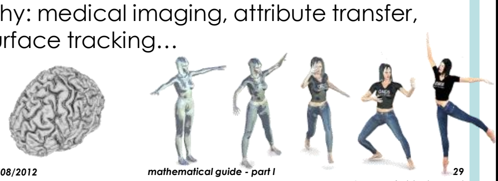
content

surface correspondence

- problem: finding correspondences between a discrete set of points on two surface meshes
  - *extrinsic* vs *intrinsic* correspondence



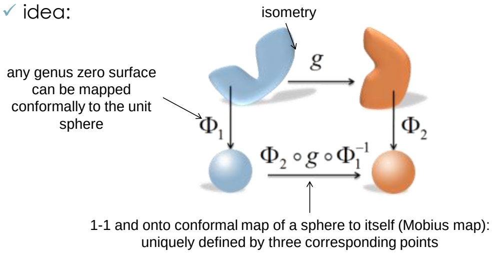
- why: medical imaging, attribute transfer, surface tracking…



Courtesy of Michael Bronstein

intrinsic correspondence [Lipman and Funkhouser 2009]

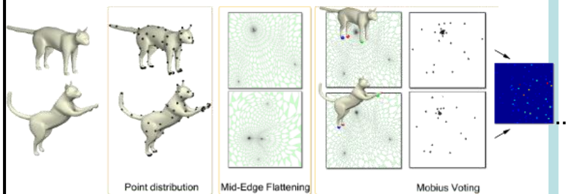
- looking for **intrinsic** correspondence means finding corresponding points such that the mapping between them is close to an **isometry**
- idea:



intrinsic correspondence [Lipman and Funkhouser 2009]

1) sampling points: local maxima of Gauss curvature & (geodesically) farthest point algorithm

3) compute the Mobius transformation that aligns a triplet in the common domain



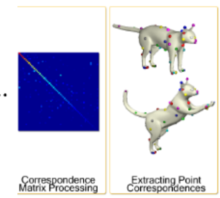
2) discrete conformal flattening to the extended complex plane

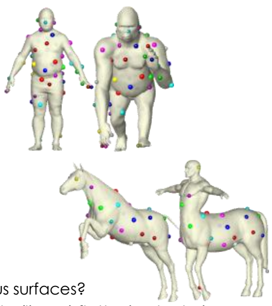
4) evaluate the intrinsic deformation error and build a fuzzy correspondence matrix

intrinsic correspondence [Lipman and Funkhouser 2009]

5) produce a discrete set of correspondences

…

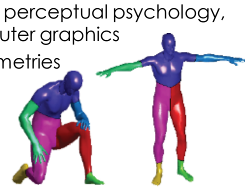




- ✓ pay attention to…
  - what about higher genus surfaces?
  - drawbacks of the discrete (linear) flattening technique

symmetry detection

- ✓ problem: detecting symmetries on a surface mesh
  - well studied problem in perceptual psychology, computer vision, computer graphics
  - *extrinsic* vs *intrinsic* symmetries



- ✓ why: compression, completion, matching, beautification, alignment…

Courtesy of Michael Bronstein

global intrinsic symmetry detection [Kim et al. 2010]

- ✓ looking for **intrinsic** symmetry transformations means finding isometric transformations that map a surface onto itself (**self-isometries**); cf. the correspondence finding problem
  - we have: M orientable, genus zero surface
  - we look for: f: $M \rightarrow M$ intrinsic symmetry
- ✓ orientation-preserving isometries are related to conformal maps; orientation-reversing isometries are related to anti-conformal maps
- ✓ ideas similar to the work just seen: here the anti-Mobius group (Mobius maps plus Mobius maps composed with a reflection) comes into play

global intrinsic symmetry detection [Kim et al. 2010]

1) sampling points: symmetry invariant sets S1 (coarse) and S2 (dense)

- ✓ IDEA: Given a symmetry invariant function Φ: $M \rightarrow$ R, the set S of its critical points is a symmetry invariant set that satisfies $f(S) = S$
- ✓ IDEA: The Average Geodesic Distance (AGD) and the Minimal Geodesic Distance (MGD) are symmetry invariant functions

$$\Phi_{agd}(p) = \int_{\mathcal{M}} d_g(p,q) d\mathrm{vol}_{\mathcal{M}}(q)$$

$$\mathcal{S}_1 = \{p \in \mathcal{M} \mid \nabla|_p \Phi_{agd} = 0\}$$

$$\Phi_{mgd}(p) = \Phi_{mgd}(p; \mathcal{S}_1) = \min_{q \in \mathcal{S}_1} d_g(p,q)$$

$$\mathcal{S}_2 = \{p \in \mathcal{M} \mid \nabla|_p \Phi_{mgd} = 0\}$$

global intrinsic symmetry detection [Kim et al. 2010]

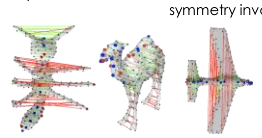
1) sampling points: symmetry invariant sets S1 (coarse) and S2 (dense)

2) discrete conformal flattening to the extended complex plane

3) compute anti-Mobius transformations that align triplets and quadruplets of S1 in the common domain

4) apply th transformations to S2 and evaluate the intrinsic deformation error (based on geodesics)

5) use the best transformation to extract correspondences within the symmetry invariant set S2

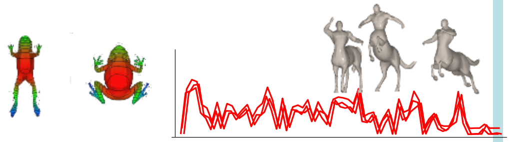


pay attention to…

- ✓ symmetry invariant functions have to be smooth!
- ✓ what about partial and small symmetries?

deformable shape characterization

- problem: describing intrinsic shape properties, and deriving signatures



Courtesy of Michael Bronstein

- why: shape registration, global and partial matching, …

intrinsic shape description [Sun et al. 2009]

- *heat equation*, governing the distribution of heat from a source on a surface X; initial conditions: heat distribution at time $t = 0$:

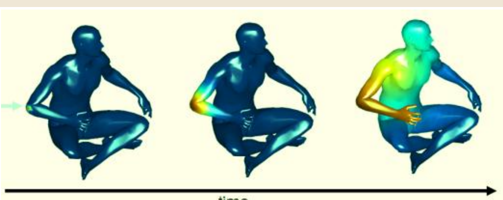
$$\left(\Delta_X - \frac{\partial}{\partial t}\right) u = 0$$

- the heat kernel $K_t(x,y)$ is a fundamental solution of the heat equation with point heat source at x (and heat value at y after time t)

$$K_t(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

$$\approx \sum_{i=0}^{N} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

intrinsic shape description [Sun et al. 2009]



$$K_t(x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

$$\approx \sum_{i=0}^{N} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

Courtesy of Michael Bronstein

intrinsic shape description [Sun et al. 2009]

- heat kernel signature as a point description over the temporal domain:

$$HKS(x): \mathbb{R}^+ \to \mathbb{R}, \qquad HKS(x,t) = k_t(x,x)$$

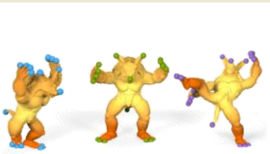
- multiscale, informative, intrinsic, localized sensitivity to topological noise

- distance between signatures at scale $[t_1, t_2]$:

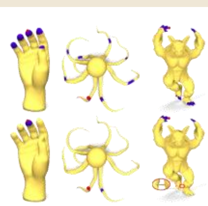
$$d_{[t_1,t_2]}(x,x') = \left(\int_{t_1}^{t_2} \left(\frac{|k_t(x,x) - k_t(x',x')|}{\int_M k_t(x,x)dx}\right)^2 d\log t\right)^{1/2}$$

Courtesy of Michael Bronstein

intrinsic shape description [Sun et al. 2009]



feature point detection



localizing repeated structures at differet scales

- pay attention to…
  - multiscale… but how to handle time scale?
  - discretization and robustness issues

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content

- mathematical concepts
  - basics on algebraic topology
  - simplicial complexes
  - homology
  - surface genus
  - critical points
  - Morse theory
- concepts in action
  - persistent topology
  - Reeb graphs

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  - Reeb graphs

about topological spaces and functions (again)

- we could think of perceptions as pairs (X, f), where X is a topological space and $f : X \to \mathbb{R}^k$ is a (continuous) function
  - X represents the set of observations made by the observer
  - for each observation $x \in X$, $f(x)$ describes x as seen by the observer
- topological spaces and continuous functions allow us to talk about stability
- so, what mathematical tools?

basics on algebraic topology

- algebraic topology associates algebraic invariants to each space so that two spaces are homeomorphic if they have the same invariants
- approach: to decompose a topological space into simple pieces that are easier to study (e.g. to decompose a polyhedron into faces, edges, vertices or a surface into triangles)

The Hitchhiker's Guide to the Galaxy of Mathematical Tools for Shape Analysis

SIGGRAPH 2012 *Course Notes*

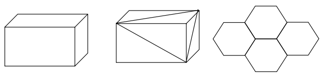
Mathematical Guide (Part 2)

basics on algebraic topology

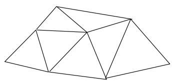
- a combinatorial structure is generated by the decomposition of the topological space
- basic elements of the decomposition are cells or simplices that are characterized by
  - combinatorial aspect: relations among the cells of the complex
  - geometric aspect: related to their embedding in the Euclidean space

examples

- cells complexes
  - a geographic map (which is made of points, lines and regions)
  - a decomposition of a polyhedron into faces

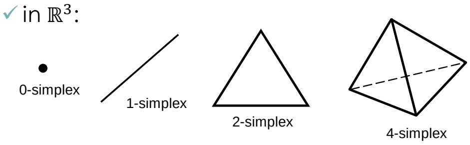


- simplicial complexes
  - e.g., triangle meshes



simplex

- a k-*simplex* in a Euclidean space $\mathbb{R}^n$, with $n \geq k$, is the convex hull of $k+1$ affinely independent points. A subset of these points defines a simplex of dimension $< k$ called *face*.
- in $\mathbb{R}^3$:



simplicial complex

- a (finite) simplicial complex K is a (finite) collection of simplices so that
  - if $\sigma \in K$ and τ is a face of σ, then $\tau \in K$
  - if $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2$ is a face of both (that is, two simplices can only meet along a common face)
- the *dimension* of K is the highest among the dimensions of its simplices
  - triangle meshes are 2-complexes
  - tetrahedral meshes are 3-complexes

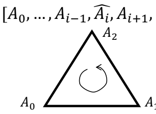
cycles and boundaries

- let K be a simplicial complex of $\mathbb{R}^n$; a q-*chain* is a formal linear combination of q- simplices
- *boundary operator*:

$$\partial_q[A_0, A_1, \dots, A_q] = \sum_{i=0}^{q} (-1)^i [A_0, \dots, A_{i-1}, \widehat{A_i}, A_{i+1}, \dots, A_q]$$

$$A_0 \longrightarrow A_1$$

$$\partial_1[A_0, A_1] = [A_0] - [A_1]$$

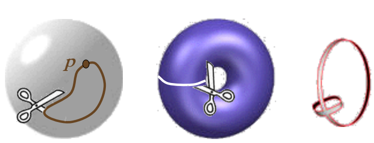


$$\partial_2[A_0, A_1, A_1] = [A_1, A_2] - [A_0, A_2] + [A_0, A_1]$$

- a chain is called a *cycle* if the boundary operator sends it to zero
- a chain is called a *boundary* if it is the image of a chain of dimension greater by 1

loops on a surface

- a *loop* (1-*cycle*) is a closed curve whose initial and final points coincide in a fixed point p known as the *basepoint*



simplicial homology

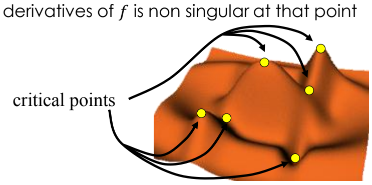
- the *q-th simplicial homology group* of K, $H_q(K)$, is the quotient group of cycles modulo boundaries
  - an element of $H_q(K)$ is an equivalence class, called *homology class*, of homologous q-cycles, that is, cycles whose difference is a boundary
- the rank of $H_q(K)$ is called the *q-th Betti number* of K, and it is a measurement of the number of different holes in K
  - for 3D data the three Betti numbers β_0, β_1, and β_2 count the number of connected components, tunnels, and voids, respectively
- the *homology* $H_*(K)$ is a topological invariant of K

genus

- the *genus* g of a surface S without boundary is:
  - half the first Betti number of S
  - the cardinality of a minimal set of mutually non-isotopic loops which can be cut along the surface without disconnecting it
- any orientable surface without boundary is a connected sum of g tori, where g is its genus, $g \geq 0$
- the genus of a surface with boundary is the genus of the surface S' obtained by gluing a disc onto each boundary component
- the genus of a surface is a topological invariant

functions and critical points

- given a smooth function $f: M \to \mathbb{R}$ on a smooth manifold M, a point x is called
  - *regular* if the differential df_x is surjective
  - *critical* if df_x is the zero map
- a critical point is called
  - *non-degenerate* if the Hessian matrix H of the second partial derivatives of f is non singular at that point



critical points

- formally:
  - a point P is critical for f if:

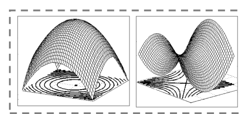
$$\frac{\partial f}{\partial x_1}(p)=0, \frac{\partial f}{\partial x_2}(p)=0, \ldots, \frac{\partial f}{\partial x_k}(p)=0$$

  - It is Morse if:

$$\left|H_f(p)\right| = \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right| \neq 0$$

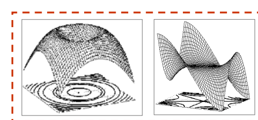
- If x is a non-degenerate critical point of f, the number λ of negative eigenvalues of H is called the index of x

critical points on a surface



non-degenerate c. p.

degenerate c. p.



Euler formula

- $\#maxima - \#saddles + \#minima = \chi(S)$
  (differential geometry)
- $2\pi(v - e + f) = K(S)$
  (differential topology)
- $v - e + f = \chi(S) = 2 - 2g$
  (algebraic topology)

Morse theory

- a function f is called *Morse* if all of its critical points are non-degenerate
- Morse theory studies the relationship between a function's critical points and the topology of its domain
- it indicates when the topological type changes and what kind of changes take place
- it provides a surface decomposition into a limited set of primitive topological cells, defined by the surface critical points and their corresponding index

does any Morse function exist?

- on any smooth compact manifold there exist Morse functions
- Morse functions are everywhere dense in the space of all smooth functions on the manifold
- any Morse function has only a finite number of critical points on a compact manifold
- the set S of all simple Morse functions is everywhere dense in the set of all Morse functions
- examples of Morse functions on a smooth manifold: height function, distance functions, geodesic distance, etc.

Morse theory & critical points

- Let C_i = #{critical points of index i} and β_i the i-th Betti number of M; then
  - Weak Morse inequalities
    - $\beta_i \leq C_i$
    - $\sum_i (-1)^i C_i = \sum_i (-1)^i \beta_i = \chi(M)$
  - Strong Morse inequalities
    - $\forall i \geq 0,\ \beta_i - \beta_{i-1} + \cdots + \beta_0 \leq C_i - C_{i-1} + \cdots + C_0$

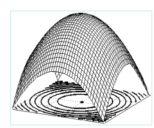
$$\#maxima - \#saddles + \#minima = \chi(M) = 2 - 2g$$

Morse theory & critical point configuration

- (Morse Lemma) In a neighbourhood of each non-degenerate critical point P, the function f can be expressed as:

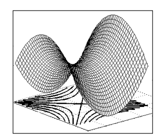
where λ is the index of the critical point

2-manifolds



$f = -x^2 - y^2$

maximum
λ=2

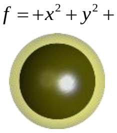


$f = -x^2 + y^2$

saddle
λ=1

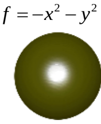
3-manifolds

$f = +x^2 + y^2 + z^2$



minimum
λ=0

$f = -x^2 - y^2 - z^2$



maximum
λ=3

$f = -x^2 + y^2 + z^2$

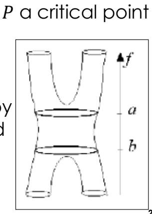
saddle
λ=1

$f = -x^2 - y^2 + z^2$

saddle
λ=2

Morse theory & critical points

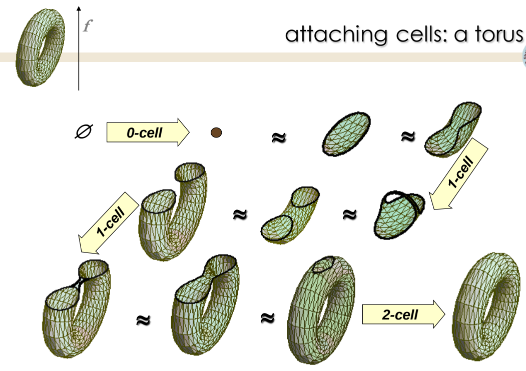
- let $f: M \to \mathbb{R}$ be a real valued function and let $[a,b] \subset \mathbb{R}$ be an interval non containing critical values of f; then the level sets $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic
- denote $M^x = \{p \in M | f(p) \leq x\}$ and P a critical point such that $f(p) = c$; then:
  $\forall \varepsilon > 0$ such that $f^{-1}[c-\varepsilon, c+\varepsilon]$ contains no other critical points of f, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached



Morse theory & shape decomposition

- Theorem (CW complex decomposition)
  - let S be a smooth compact manifold embedded in an Euclidean space. Let f: $S \to \mathbb{R}$ be a smooth, real valued, Morse function on S. Then S is homeomorphic (i.e. topologically equivalent) to a cell complex of dimension i for each critical point of index i

attaching cells: a torus



pictures from http://www.cs.rug.nl/~gert/topology.html

Morse theory does not say

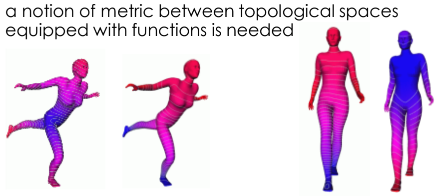
- that all smooth functions on S have the same number of critical points
- if the cell complex obtained using a given f is the "best possible" (i.e. it has the fewest number of cells)

homology, Morse theory, and shape description

- medial axis transform *(Blum 1967)*
- shock graphs *(Kimia,Tannenbaum, Zucker 1995)*
- surface networks *(Pfaltz 1976)*
- skeletons and centerlines *(Sethian 1985, Bloomenthal 1991)*
- apparent contours *(Haefliger 1960, Pignoni 1991)*
- size functions *(Ferri, Frosini 1990)*
- barcodes *(Zomorodian et al 2004)*
- Reeb graphs *(Reeb 1946)* & contour trees *(Boyell & Ruston1963)*

about distances (again)

- to assess how far two perceptions are
  - a notion of metric between topological spaces equipped with functions is needed



- natural pseudo-distance: shapes are similar if there exists a homeomorphism between the spaces that preserves the properties conveyed by the functions

natural pseudo-distance

- let H be a (subset of) the set of all homeomorphisms $\gamma: X \to Y$, the natural size pseudodistance $\delta((X,\phi),(Y,\psi))$ is
- $\delta\big((X,\phi),(Y,\psi)\big) = f(x) = \begin{cases} inf_{\gamma \in H}\Theta(\gamma), & H \neq \emptyset \\ +\infty, & H = \emptyset \end{cases}$

$$\Theta(\gamma) = \max_{P \in X} \|\phi(P) - \psi(\gamma(P))\|_\infty$$

- δ is small iff $\exists\gamma$ that induces a small change on the function f, that is, if there exist an homeomorphism mapping one space into the other while preserving the properties conveyed by the real function
- how to compute it? Stay tuned...

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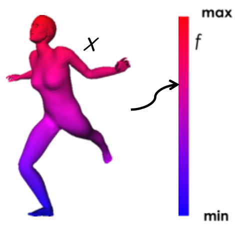
content

- mathematical concepts
  - basics on algebraic topology
  - simplicial complexes
  - loops on a surface
  - homology
  - surface genus
  - critical points
  - Morse theory
- concepts in action
  - persistent topology
  - Reeb graphs

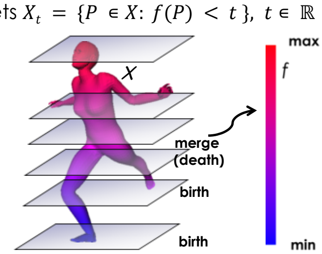
Morse Theory and computational topology



a *shape* is a pair (X, f), that is, a topological space X endowed with a real function $f: X \rightarrow \mathbb{R}$ describing its properties

persistent topology

- topological events *[e.g. birth, merge of connected components]* occur while sweeping X through f, i.e., when analysing X as the evolution of its sub-level sets $X_t = \{P \in X: f(P) < t\}, t \in \mathbb{R}$



persistent topology

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basic idea:
encode the lifespan of topological events; lifespan is proportional to the importance of the features they represent: long events stand for significant features, short events stand for either details or noise

an historical perspective

- different topological features:
  - size functions: [Frosini 1991], 0-th degree homology (connected components)
  - persistent homology: [Edelsbrunner et al. 2000], higher degree homology (cycles)
- from 1-dimensional to multidimensional properties:
  - 1-dimensional setting: $f: X \rightarrow \mathbb{R}$ [Frosini 1991, Edelsbrunner et al. 2000]
  - multidimensional setting: $\vec{f}: X \rightarrow \mathbb{R}^k$ [Biasotti et al. 2008]

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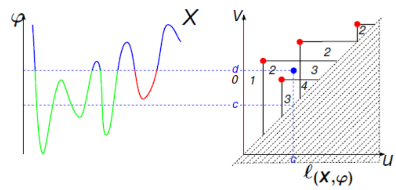
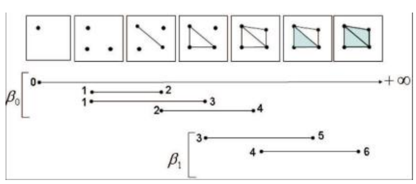
size functions

- the size function $\ell_{(X,\varphi)}: \{(u,v) \in \mathbb{R}^2: u < v\} \to N$ takes each $(u,v)$to the number of connected components of X_v that contain at least one point of X_v
  - that is,$\ell_{(X,\varphi)}(u,v)$ = number of connected components (0-homology classes) born before u and still alive at v



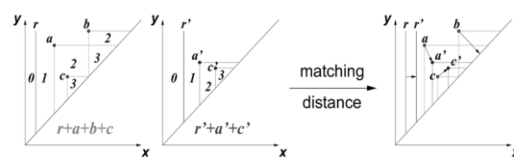
persistent homology (barcodes)

- given the pair (X,f), consider the collection of nested lower level sets of f: $X(f \preccurlyeq u) = \{x \in X : f(x) \preccurlyeq u\}$;
- measure the scale at which a topological feature (e.g., a connected component, a tunnel, a void) is created, and when it is annihilated along this filtration using homology groups
- encode this information as parametrized Betti numbers



size functions and matching distance

- the matching distance between size functions is stable wrt noise and approximates the natural pseudo-distance



size functions and matching distance

- stability theorem:

$$\max_{P \in \mathcal{M}} |\varphi(P) - \psi(P)| \leq \epsilon \Rightarrow d_{match}(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{M},\psi)}) \leq \epsilon.$$

small changes in the measuring functions imply small changes in the size functions: robustness wrt perturbation of the data

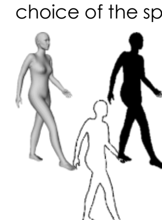
- lower bound for the natural pseudo-distance:
  Let λ be the value of the matching distance between the two size functions $\ell_{(M,\varphi)}$ e $\ell_{(N,\psi)}$. Then

$$d((M,\varphi),(N,\psi)) \geq \lambda.$$

there is a link between the comparison of size functions and the comparison of shapes

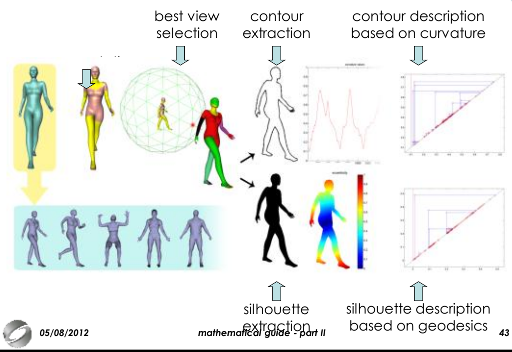
modularity

- parametric wrt the choice of the space X



- parametric wrt the choice of the function f

size functions and view-based 3D retrieval [Mortara et al. 2010]



an historical perspective

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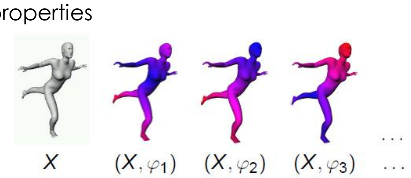
  - multidimensional setting: $\vec{\varphi}: X \rightarrow \mathbb{R}^k$ [Biasotti et al. 2008]

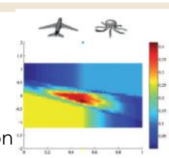
multidimensional setting

- ✓ why a vector-valued measuring function?
- ✓ many properties are intrinsically multidimensional *(coordinates, colour...)*; alternatively, we may want to blend the information of different one-dimensional properties

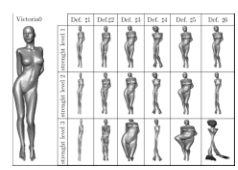


multidimensional setting: (still) open issues

- ✓ what is at stake
  - think of an algorithm to compute such stuff [Biasotti et al. 2011]
  - and test it in shape comparison

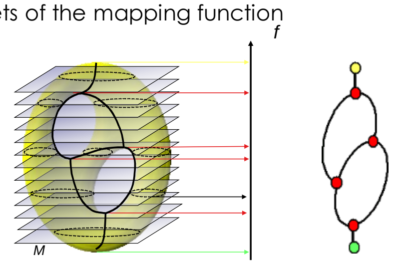


Dmatch						
	0.00	1.59	7.77	7.77	23.54	23.99
	1.59	0.00	7.77	7.77	24.71	25.15
	7.77	7.77	0.00	3.42	22.63	23.08
	7.77	7.77	3.42	0.00	20.07	20.51
	23.54	24.71	22.63	20.07	0.00	1.25
	23.99	25.15	23.08	20.51	1.25	0.00



Reeb graph

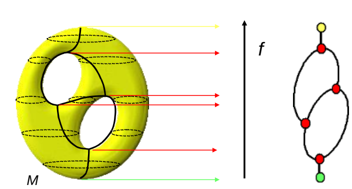
- ✓ Reeb graphs store the evolution of the level sets of the mapping function



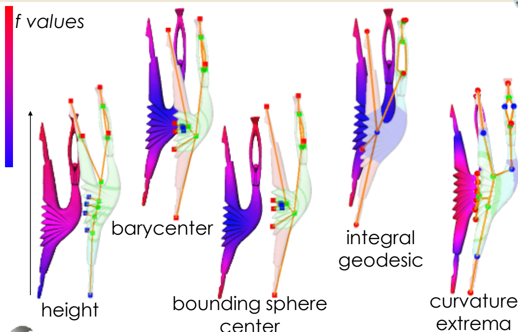
Reeb graph definition

- ✓ given f: $S \rightarrow \mathbb{R}$ defined on the surface S, the Reeb graph of S wrt f is the quotient space defined by "~" :

$(x_1, f(x_1)) \sim (x_2, f(x_2)) \Leftrightarrow f(x_1) = f(x_2)$ & x_1, x_2 are in the same connected component of $f^{-1}(f(x_1))$

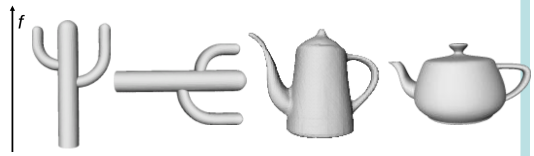


overview of RGs when the function *f* varies

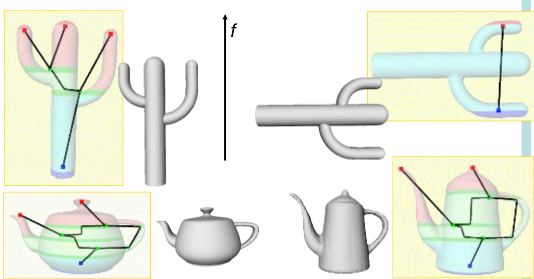


quiz

- Draw the Reeb graph with respect to the height function f of the following shapes

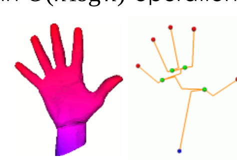


quiz - solutions



RG properties

- it provides a 1D structure of the shape
- it describes the shape of an object under the lens of the function f
- nodes and arcs depend on the number of critical points of f
- it may be computed in $O(n \log n)$ operations

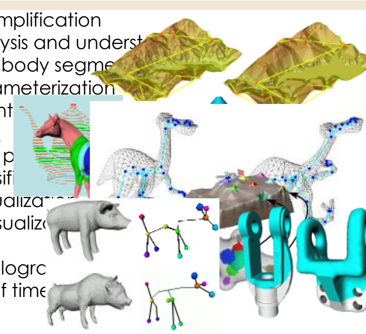


Reeb graph based representations

- Reeb graph variations
  - contour trees (simply-connected domains)
  - component trees (gray-level images)
  - centerline skeletons (geodesic distance from a point)
- for shape matching
  - Multiresolution Reeb graph (MRG), Hilaga et al. 2001
  - augmented Multiresolution Reeb graph (aMRG), Tung&Schmitt 2005
  - Extended Reeb graph (ERG), Biasotti et al. 2000

applications

- topology simplification
- shape analysis and underst
- shape and body segme
- shape parameterization
- handle ident
- animation
- global and p
- shape classif
- volume visualiza
- scientific visualiz
- rendering
- X-ray cristallogra
- analysing of time



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The Hitchhiker's Guide to the Galaxy of Mathematical Tools for Shape Analysis

SIGGRAPH 2012 Course Notes

Discussions and open issues

getting out of a maze

- complex shapes, complex problems and… mathematical concepts…
  - *what is the only maths you can't ever apply? The one you don't know!*
    (Mario Pezzana, by way of Massimo Ferri*)*
- we had a look at mathematical theories and techniques for shape analysis, description and similarity
- but if we go back to shapes and problems, are we sure that everything works ok?
- let's start by having a look to «real» world

complex shapes, complex problems

- online repositories of 3D models
  - Google 3D Warehouse http://sketchup.google.com/3dwarehouse/

complex shapes, complex problems

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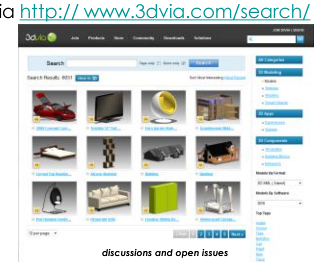
  - 3Dvia http:// www.3dvia.com/search/



complex shapes, complex problems

- online repositories of 3D models
  - Google 3D Warehouse http://sketchup.google.com/3dwarehouse/
  - 3Dvia http:// www.3dvia.com/search/
  - Turbosquid http://www.turbosquid.com/
    "…search our stock catalog to get the 3D model you want, or use our Custom 3D modeling service for made-to-order 3D models. Join the world's top artists who use TurboSquid 3D models in advertising, architecture, broadcast, games, training, film, the web, and just for fun"

complex shapes, complex problems

- Digital Shape Workbench v5.0 http://visionair.ge.imati.cnr.it/
  - an infrastructure offering shapes and software tools

questions&answers

- do these approaches meet the requirements of the mathematical methods presented?
- when and how is robustness *really* guaranteed?
- under what conditions do methods really work?
- theoretical answers and benchmarking

to sum up: theory says …

- 3D shapes maybe very complex and there are many concepts and tools, so…
- … how to get out of this maze?
- basics
  - the choice of a shape description despite another depends on
    - type of shapes and their variability/complexity
    - invariants or properties
  - topological spaces and functions are necessary to model the observer's perception and to reason to reason about stability and perturbation

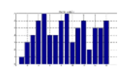
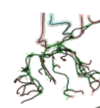
pay attention to…

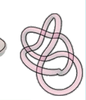
- … the right metric space
  - rigid transformations (rotations, translations)
    - Euclidean distances
  - isometries
    - Riemannian metric
    - curvature (but unstable to local noise/perturbations)
    - geodesics, diffusion geometry, Laplacian operators, etc
  - local invariance to shape parameterizations
    - conformal geometry
  - similarities (i.e. scale operations)
    - normalized Euclidean distances
  - affinity (and homeomorphisms)
    - persistent topology
    - Morse theory
    - size theory



pay attention to…

- … to a suitable shape description
  - coarse coding (but fast)
    - histograms
    - matrices
  - articulated shapes
    - medial axes
    - Reeb graphs
  - overall global appearance
    - silhouettes
  - if shape loops are relevant
    - persistent topology
    - graph-based descriptions




to sum up: benchmarking

- SHape Retrieval Context
  http://www.aimatshape.net/event/SHREC
  - an annual event to to evaluate the effectiveness of 3D shape analysis algorithms
  - a multi-track event spanning
    - different models: from watertight objects to triangle soups, from abstract shapes to medical data
    - different tasks: from 3D retrieval to correspondence finding and segmentation

DEMO

acknowledgements

- ***VISIONAIR:*** *Vision Advanced Infrastructure for Research*, European project "FP7 INFRASTRUCTURES", 2011-2015
- ***MULTISCALEHUMAN:*** *Multi-scale Biological Modalities for Physiological Human Articulation*, European project "FP7 PEOPLE" Initial Training Network, 2011-2014
- P. Frosini. M. Ferri and the *Vision Mathematics group* at the Dept. of Mathematics, University of Bologna
- C. Landi, Dept. of Science and Methods of Engineering, University of Modena and Reggio Emilia

at the end of the day…

SIGGRAPH 2012 *Course Notes*

Thank you for your attention!